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Nonlinear evolution equations of Hamiltonian form and the Painlevé analysis

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Abstract. A number of necessary conditions for a class of nonlinear partial differential equations to pass the Painlevé test with the *Kruskal* ansatz is given. A theory for the admissible resonance patterns of equations in Hamiltonian form is developed and discussed for some important special cases. Based on these results an algorithm can be described, which constructs *all* nonlinear evolution equations of normal type and a certain Hamiltonian form.

1. Introduction

Painlevé type equations, i.e. those whose solutions have no movable critical points [9], have recently become popular due to their connection with partial differential equations integrable by the inverse scattering transform [1]. In [2] an algorithm to test whether a given ordinary differential equation satisfies necessary conditions for it to be of Painlevé type is presented. In [14] a computer algebra package using MACSYMA and based on a modification of this algorithm is described. However, there is a problem as the system cannot handle free parameters automatically, if they affect the resonances. In [18] a test which uses partial differential equations directly is proposed and [3] introduces a simplified version. In [8, 10, 11, 17, 13] the problem of classifying differential equations based on the requirements of the Painlevé property is further clarified. In [15] the theory of an algorithm and a REDUCE package to classify *all* nonlinear evolution equations of normal type (and a further restriction on the order) containing free real parameters based on a combination of an improvement of the approach developed in [7] and results of the previously cited literature are described. The crucial step in this procedure is to identify all resonance polynomials a certain class of evolution equations can have at most, if its members should pass the Painlevé test.

There is one practical problem arising from this—the number of possible resonance patterns to be investigated increases very rapidly with the *degree of homogeneity* (see next section). If we make the stronger restriction that the evolution equations under consideration are not only normal but also of Hamiltonian form, the number of possible resonance patterns in the Painlevé case decreases drastically because of a phenomenon called *resonance pairing* first observed in the context of *similarity invariant systems* of ordinary differential equations [19]. In [12] a generalization of this result is given for ODEs of Hamiltonian form analogous to lemma 6 in this paper. In [6] the authors discussed the occurrence of pairs of resonances in Hamiltonian ODEs from a geometrical point of view.

2. Definitions and fundamentals

We study scalar nonlinear evolution equations

$$\Delta : u_t - K[u] = 0$$

with $u = u(x,t)$ and $K[u] \in \mathcal{P}$, where $\mathcal{P}$ denotes the vector space of differential polynomials with respect to the independent variable x. $K[u]$ consists of terms of the form

$$T = \alpha \prod_{c=0,\dots,k} u_{cx}^{a_c} \qquad u_{jx} := \frac{\partial^j u}{\partial x^j} \qquad a_c \in \mathbb{N}_0 \qquad \alpha \in \mathbb{R}.$$

We call such a term a *monomial.* The *order* of $K[u]$, $o(K)$, is defined as

$$o(K) := \max\left\{ k \middle| \frac{\partial K}{\partial u_{kx}} \neq 0 \right\}.$$

Further we assume that $K[u]$ is of normal form, i.e. $K[u] = u_{mx} + \tilde{K}[u]$ with $o(\tilde{K}) < m$.

We call K *homogeneous of degree* h iff there is a *weight* $w \in \mathbb{Q}$, so that for all monomials T_i in K

$$\sum_{c_i=0}^{k_i} a_{c_i}(c_i - w) = h$$

is satisfied and we write $K \in \mathcal{H}_w^h$.

Example 1. (KdV) $\Delta : u_t - (u_{3x} + 12uu_x) = 0$, $K \in \mathcal{H}_{-2}^5$, since with the weight $w = -2$ we get the degree of homogeneity $1(3+2) = 5$ for u_{3x} and uu_x is homogeneous of the same degree because $1(0+2) + 1(1+2) = 5$. Adding $u_{2x} \in \mathcal{H}_{-2}^4$, for example, makes $K[u]$ inhomogeneous.

Remark. For a given weight w it is clear that $\mathcal{H}_w^h$ is a subspace of $\mathcal{P}$ and

$$\mathcal{P} = \bigoplus_{h \in \mathbb{Q}} \mathcal{H}_w^h. \tag{1}$$

is valid.

The differential operator (DO) $D_x : \mathcal{P} \to \mathcal{P}$ is defined in the form

$$D_x := \sum_{k=0}^{\infty} u_{(k+1)x} \frac{\partial}{\partial u_{kx}}.$$

Given a polynomial DO (pDO) $A[u] : \mathcal{P} \to \mathcal{P}$ of the form

$$A[u] = \sum_{\nu=0}^{n} P_\nu[u] D_x^\nu \qquad P_\nu \in \mathcal{P} \cup \mathbb{R}$$

then the operator

$$A^*[u] := \sum_{\nu=0}^{n}(-1)^{\nu} D_x^{\nu} P_{\nu}[u]$$

is called the *adjoint* operator of A.

A pDO $J[u] : \mathcal{P} \to \mathcal{P}$ is called *skew-adjoint* and *self-adjoint*, respectively, iff

$$J^*[u] = -J[u] \qquad J^*[u] = J[u].$$

Remark. Every (finite) skew-adjoint pDO $J[u] : \mathcal{P} \to \mathcal{P}$ can be written in the form

$$J[u] = \sum_{\nu=0}^{n} P_{\nu}[u] D_x^{\nu} + (-1)^{\nu+1} D_x^{\nu} P_{\nu}[u] \qquad P_{\nu} \in \mathcal{P} \cup \mathbb{R}. \tag{2}$$

The pDO $\delta : \mathcal{P} \to \mathcal{P}$ with

$$\delta := \sum_{\nu=0}^{\infty}(-1)^{\nu} D_x^{\nu} \frac{\partial}{\partial u_{\nu x}} \tag{3}$$

is called a *variational derivative.*

An evolution equation

$$\Delta : u_t - J[u]\delta H[u] = 0 \tag{4}$$

with $J[u]$ skew-adjoint is said to be of *Hamiltonian form.*

Since we assume $K[u]$ to be normal, H has to be of the form $H[u] = (-1)^k \frac{1}{2} u_{kx}^2 + \tilde{H}[u]$ with $o(\tilde{H}) < k$ and n odd with $P_n = \frac{1}{2}$ in (2).

Remark. With the Noether theorem $H[u]$ is a *conserved density.* H can be given in an unambiguous way using the algorithm described in [5]. There H is called *irreducible* iff in every monomial of $H[u]$ the highest derivative of u is nonlinear, e.g. uu_x is not an irreducible monomial whereas u^2 and u_x^2 are.

According to the definition of a degree of homogeneity given above, a pDO $A[u] : \mathcal{P} \to \mathcal{P}$ of the form

$$A[u] = \sum_{\nu=0}^{n} P_{\nu}[u] D_x^{\nu} + D_x^{\nu} Q_{\nu}[u]$$

can also be assigned a degree of homogeneity $h \geqslant n$ iff $P_{\nu}, Q_{\nu} \in \mathcal{H}_w^{h-\nu}$ or $P_n, Q_n \in \mathbb{R}$ for $n = h$. The variational derivative δ is homogeneous of the degree $h = w$.

If an evolution equation Δ with $o(K) = m$ passes the Painlevé test proposed in [18] in the *Kruskal* modified form [3], also called the *resonance form*, then there exists an expansion

$$u = \xi^{-l} \sum_{\nu=0}^{\infty} u_{\nu}(t)\xi^{\nu} \qquad \xi = x + f(t) \qquad l \in \mathbb{N} \tag{5}$$

which solves Δ formally and provides $m-1$ free functions $u_\nu(t)$. l is called the *leading order*. We say Δ is a *Painlevé property candidate* (PPC) [15], iff at least one expansion of this form exists.

Assuming $K[u] \in \mathcal{H}^h_{-l}$ and inserting (5) into Δ, we get

$$\Delta : u_t - K[u] = \sum_{j=0}^{\infty} (T_\nu - H_\nu)\, \xi^{\nu-h} = 0$$

where (with $u_i = 0$ for $i < 0$)

$$T_j = 0 \qquad \text{for } 0 \leqslant j < h - l - 1$$

or

$$T_j = (j - h + 1)\dot{f} u_{j-(h-l)+1} + \dot{u}_{j-(h-l)}$$

and

$$H_n = P_n(u_0)u_n - C_n(u_0, \ldots, u_{n-1}).$$

Altogether we have to solve the (infinite) system of equations

$$P_0(\alpha)\alpha\xi^{-h} = K[\alpha\xi^{-l}] = 0 \tag{6}$$

with solution(s) $\alpha = u_0$ and

$$P_n(u_0)u_n = C_n(u_0, \ldots, u_{n-1}) + T_n(u_{n-(h-l)+1}, \dot{u}_{n-(h-l)}, \dot{f}). \tag{7}$$

Remark. To avoid T_n depending on u_j with $j \geqslant n$ in (7), we assume $h > l+1$ in the following. This restriction only cancels trivial evolution equations as $\Delta : u_t - u_x = 0$.

Because (7) is linear in u_n, we obtain its left-hand side with the help of the *Fréchet derivative*:

$$D_K[u]Q := \frac{\partial}{\partial \varepsilon} K[u + \varepsilon Q]|_{\varepsilon=0} = \sum_{\nu=0}^{\infty} \frac{\partial K[u]}{\partial u_{\nu x}} D_x^\nu Q$$

in the form

$$D_K[\alpha\xi^{-l}]u_n\xi^{n-l}|_{\xi=1} = P_n(\alpha)u_n =: Q(\alpha, n)u_n. \tag{8}$$

$u_0 \neq 0$ with $P_0(u_0)u_0 = 0$ is called a *branch*, r with $Q(u_0, r) = 0$ *resonance* (of the branch u_0). If the context is clear, we omit the extension *in the branch* u_0.

The pDO $D_{\delta B}[u]$ is self-adjoint for every $B \in \mathcal{P}$, since

$$\begin{aligned}
D_{\delta B}[u] &= \sum_{\nu=0}^{\infty} \frac{\partial(\sum_{k=0}^{\infty}(-1)^k D_x^k \partial B[u]/\partial u_{kx})}{\partial u_{\nu x}} D_x^\nu \\
&= \sum_{\nu=0}^{\infty}\sum_{k=0}^{\infty} (-1)^k D_x^k \frac{\partial^2 B[u]}{\partial u_{kx}\partial u_{\nu x}} D_x^\nu \\
&\overset{\nu \leftrightarrow k}{=} \sum_{k=0}^{\infty}\sum_{\nu=0}^{\infty} (-1)^k(-1)^k(-1)^\nu D_x^\nu \frac{\partial^2 B[u]}{\partial u_{\nu x}\partial u_{kx}} D_x^k \\
&= D^*_{\delta B}[u].
\end{aligned} \tag{9}$$

3. Necessary conditions for PPCs

When the resonance r is a non-negative integer we see that the left-hand side of (7) vanishes for $n = r$. This means that u_r cannot be determined from this equation, but the right-hand side consists of previously determined functions u_i. If this right-hand side also vanishes, then we say the *compatibility condition* is fulfilled otherwise Δ is not a PPC.

As a consequence of the demand for $m - 1$ free functions $u_i(t)$ in the PPC case we get the following lemma.

Lemma 1. The resonance polynomial $Q(u_0, r)$ must have $m - 1$ different non-negative integer zeros in at least one branch u_0.

We call this (these) branch(es) the *principal branch(es)* [13].

With the help of the normalization $u \to -u_0 u$ one of these principal branches u_0 always equals -1.

In [17] the following lemma has been proven.

Lemma 2. $r = -1$ is a resonance in every branch u_0.

Remark. The reader should note that for equations which are *not* of normal form this lemma does *not* hold in general. For examples see the cases discussed in [4].

From [15] we get

Lemma 3. If $\Delta : u_t - K[u] = 0$ is a PPC with respect to leading order l and K is homogeneous of degree h, then for the resonances r_i the following holds

$$|r_i - r_j| \neq h - l \qquad i, j \in \{1, \ldots, m\} \tag{10}$$

and from [10]

Lemma 4. If $\Delta : u_t - K[u] = 0$ is a PPC with leading order l, then $l \in \{1, 2\}$.

Remark. u_0 cannot be a free function for Δ, i.e. $r = 0$ cannot be a resonance in the Painlevé case, because we assume K to be of normal form.

If we now regard an evolution equation $\Delta : u_t - K[u] = 0$ with not necessarily homogeneous K with respect to leading order l, we can write it with (1) in the form

$$K = K_{h_1} + K_{h_2} + \cdots + K_{h_k}$$

with $K_{h_i} \in \mathcal{H}^{h_i}_{-l}$ and $h_i < h_j$ for $i > j$.

In [15] a proof is given for

Lemma 5. If $\Delta : u_t - K[u] = 0$ with $o(K) = m$, $K[u]$ not necessarily homogeneous, is a PPC with respect to leading order l and for $\tilde{\Delta} : u_t - K_{h_1}[u] = 0$ it holds that $o(K_{h_1}) = m$, then $\tilde{\Delta}$ is also a PPC with respect to leading order l.

Remark. A consequence of lemma 5 is that if a homogeneous evolution equation does not pass the Painlevé test, then any inhomogeneous extension of it cannot pass the test. So this result allows us, if we are looking for all possible evolution equations of normal form passing the Painlevé test, to start with homogeneous equations, identify the PPC cases, perform the whole test, add all monomials with lower degrees of homogeneity with free coefficients and check for which adaptation the inhomogeneous equation passes the test.

4. Resonance patterns and Hamiltonian form

In this section we uncover the reflection of the skew (or the self-) adjointness of the differential operators involved, in particular the symmetry properties of the corresponding resonance polynomials. In the following we assume Δ to be homogeneous in Hamiltonian form (4) with $H \in \mathcal{H}^{h_H}_{-l}$ and J with degree of homogeneity h_J. Then (6) becomes

$$\begin{aligned} P_0(\alpha)\alpha\xi^{-(h_H+h_J-l)} &= K[\alpha\xi^{-l}] \\ &= J[\alpha\xi^{-l}]\delta H[\alpha\xi^{-l}] \\ &=: \tilde{P}(\alpha)\hat{P}(\alpha)\alpha\xi^{-(h_H+h_J-l)} = 0. \end{aligned} \tag{11}$$

From (11) we see that there are two possibilities for a branch $\alpha = u_0$: u_0 is a branch in $\delta H[u]$, i.e. $\hat{P}(u_0) = 0$; or it is generated by $J[u]$, i.e. $\tilde{P}(u_0) = 0$. With (8) the resonance polynomial has the form

$$\begin{aligned} Q(\alpha, r) &= D_K[\alpha\xi^{-l}]\xi^{r-l}|_{\xi=1} \\ &= D_{J\delta H}[\alpha\xi^{-l}]\xi^{r-l}|_{\xi=1} \\ &= \left(D_J[\alpha\xi^{-l}]\xi^{r-l}\right)\delta H[\alpha\xi^{-l}]|_{\xi=1} \\ &\quad + J[\alpha\xi^{-l}]\xi^{r-(h_H-l)}|_{\xi=1}D_{\delta H}[\alpha\xi^{-l}]\xi^{r-l}|_{\xi=1} \end{aligned} \tag{12}$$

and it holds that $o(\delta H) = h_H - 2l$.

Example 2. *(Caudrey–Dodd–Gibbon–Sawada–Kotera)*

$$\Delta : u_t - (u_{5x} + 30uu_{3x} + 30u_xu_{2x} + 180u^2u_x) = 0$$

$$H[u] = -\tfrac{1}{2}u_x^2 + u^3 \in \mathcal{H}^6_{-2} \qquad \delta H = u_{2x} + 3u^2 \qquad J[u] = D_x^3 + 12D_xu + 12uD_x$$

$$P_0(\alpha)\alpha = K[\alpha\xi^{-2}]|_{\xi=1} = -360(\alpha^2 + 3\alpha + 2)\alpha = -360(\alpha + 1)(\alpha + 2)\alpha$$

$$\begin{aligned} \tilde{P}(\alpha)\hat{P}(\alpha)\alpha &= \left(D_x^3 + 12\alpha D_x\xi^{-2} + 12\alpha\xi^{-2}D_x\right)\xi^{-4}|_{\xi=1}(6 + 3\alpha)\alpha \\ &= -(120 + 120\alpha)3(2 + \alpha)\alpha = P_0(\alpha)\alpha. \end{aligned}$$

For the resonance polynomial we get with (12)

$$\begin{aligned} Q(\alpha, r) &= 3(12D_x\xi^{r-2} + 12\xi^{r-2}D_x)\xi^{-4}(2 + \alpha)\alpha|_{\xi=1} + ((r-4)(r-5)(r-6) \\ &\quad + 12\alpha(r-6) + 12\alpha(r-4))((r-2)(r-3) + 6\alpha) \\ &= 36(r-10)(2+\alpha)\alpha + (r-5)(r^2 - 10r + 24 + 24\alpha)(r^2 - 5r + 6 + 6\alpha) \end{aligned}$$

$\Rightarrow$

$$Q(-2,r) = (r+2)(r+1)(r-5)(r-6)(r-12)$$
$$Q(-1,r) = (r+1)(r-2)(r-3)(r-6)(r-10).$$

For the part $\hat{Q}(\alpha,r)$ of the principal resonance polynomial (12) determined by the density H we get the following symmetry property.

Lemma 6.

$$\hat{Q}(\alpha,r) := D_{\delta H}[\alpha\xi^{-l}]\xi^{r-l}|_{\xi=1} = \hat{Q}(\alpha,h_H-1-r). \tag{13}$$

Proof.

$$\begin{aligned}\hat{Q}(\alpha,r) &= D_{\delta H}[\alpha\xi^{-l}]\xi^{r-l}|_{\xi=1}\\ &\overset{(9)}{=} \sum_{\nu=0}^{\infty}\sum_{k=0}^{\infty}(-1)^k D_x^k \frac{\partial^2 H[\alpha\xi^{-l}]}{\partial u_{kx}\partial u_{\nu x}} D_x^\nu \xi^{r-l}\Big|_{\xi=1}\\ &= \sum_{\nu=0}^{\infty}\sum_{k=0}^{\infty}(-1)^k \frac{\partial^2 H[\alpha\xi^{-l}]}{\partial u_{kx}\partial u_{\nu x}}\Big|_{\xi=1} D_x^k \xi^{-(h_H-\nu-k-2l)} D_x^\nu \xi^{r-l}|_{\xi=1}\\ &= \sum_{\nu=0}^{\infty}\sum_{k=0}^{\infty}(-1)^k \frac{\partial^2 H[\alpha\xi^{-l}]}{\partial u_{kx}\partial u_{\nu x}}\Big|_{\xi=1} \prod_{j=1}^{k}(r-(h_H-l-j)) \prod_{j=1}^{\nu}(r-(l+j-1))\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}&\hat{Q}(\alpha,h_H-1-r)\\ &\quad= \sum_{\nu=0}^{\infty}\sum_{k=0}^{\infty}(-1)^k \frac{\partial^2 H[\alpha\xi^{-l}]}{\partial u_{kx}\partial u_{\nu x}}\Big|_{\xi=1}\\ &\qquad\times \prod_{j=1}^{k}(h_H-1-r-(h_H-l-j)) \prod_{j=1}^{\nu}(h_H-1-r-(l+j-1))\\ &\quad= \sum_{k=0}^{\infty}\sum_{\nu=0}^{\infty}(-1)^\nu(-1)^k(-1)^k \frac{\partial^2 H[\alpha\xi^{-l}]}{\partial u_{\nu x}\partial u_{kx}}\Big|_{\xi=1}\\ &\qquad\times \prod_{j=1}^{k}(r-(l+j-1)) \prod_{j=1}^{\nu}(r-(h_H-l-j)) \overset{k\rightleftharpoons\nu}{=} \hat{Q}(\alpha,r).\end{aligned}$$

□

As a consequence all zeros of $\hat{Q}(\alpha,r)$ are lying symmetrical to $(h_H-1)/2$ in every branch $\alpha = u_0$.

Example 3. *(Fordy–Gibbons II)*

$$\Delta : u_t - D_x\delta H[u]$$
$$= u_t - D_x(u_{6x} - 7u_x u_{4x} - 7u^2 u_{4x} - 14u_{2x}u_{3x} - 28uu_x u_{3x} - 21uu_{2x}^2$$
$$- 28u_x^2 u_{2x} + 14u^2 u_x u_{2x} + 14u^4 u_{2x} + \tfrac{28}{3}uu_x^3 + 28u^3 u_x^2 - \tfrac{4}{3}u^7) = 0$$

with

$$H[u] = -\tfrac{1}{2}u_{3x}^2 - \tfrac{7}{2}u_x u_{2x}^2 - \tfrac{7}{2}u^2 u_{2x}^2 + \tfrac{7}{6}u_x^4 - \tfrac{7}{3}u^2 u_x^3 - u^4 u_x^2 - \tfrac{1}{6}u^8 \in \mathcal{H}_{-1}^8$$

and branch polynomial

$$P_0(\alpha)\alpha = -\tfrac{28}{3}(\alpha+1)(\alpha-2)(\alpha+3)(\alpha-3)(\alpha-5)(\alpha+6)\alpha.$$

For the resonance polynomial $Q(u_0, r) = D_x\xi^{r-7}|_{\xi=1}\hat{Q}(u_0, r)$ we get

$$Q(-1,r) = Q(2,r) = (r+1)(r-2)(r-3)(r-4)(r-5)(r-7)(r-8)$$
$$Q(-3,r) = (r+5)(r+1)(r-3)(r-4)(r-7)(r-8)(r-12)$$
$$Q(3,r) = (r+2)(r+1)(r-3)(r-4)(r-7)(r-8)(r-9)$$
$$Q(5,r) = (r+5)(r+4)(r+1)(r-7)(r-8)(r-11)(r-12)$$
$$Q(-6,r) = (r+11)(r+5)(r+1)(r-7)(r-8)(r-12)(r-18)$$

with all the zeros (except $r = 7$ generated by $J[u] = D_x$) lying symmetrically to $(h_H - 1)/2 = \frac{7}{2}$.

For the part $\tilde{Q}(\alpha, r)$ of the principal resonance polynomial (12) determined by the skew-adjoint operator J we get the following symmetry property.

Lemma 7.

$$\tilde{Q}(\alpha,r) := J[\alpha\xi^{-l}]\xi^{r-(h_H-l)}|_{\xi=1}$$
$$= -\tilde{Q}(\alpha, 2(h_H - l) + h_J - 1 - r). \tag{14}$$

Proof.

$$\tilde{Q}(\alpha,r) = J[\alpha\xi^{-l}]\xi^{r-(h_H-l)}|_{\xi=1}$$
$$\overset{(2)}{=} \sum_{\nu=0}^{n} P_\nu[\alpha\xi^{-l}]D_x^\nu\xi^{r-(h_H-l)}|_{\xi=1} + (-1)^{\nu+1}D_x^\nu P_\nu[\alpha\xi^{-l}]\xi^{r-(h_H-l)}|_{\xi=1}$$
$$= \sum_{\nu=0}^{n} P_\nu[\alpha\xi^{-l}]|_{\xi=1}\prod_{j=1}^{\nu}(r-(h_H-l+j-1))$$
$$+ (-1)^{\nu+1}P_\nu[\alpha\xi^{-l}]|_{\xi=1}\prod_{j=1}^{\nu}(r-(h_H+h_J-l-j))$$

$\Rightarrow$

$$\begin{aligned}
\tilde{Q}(\alpha, 2(h_H - l) + h_J - 1 - r) \\
&= \sum_{\nu=0}^{n} \Big(P_\nu[\alpha\xi^{-l}]|_{\xi=1} \prod_{j=1}^{\nu} (2(h_H - l) + h_J - 1 - r - (h_H - l + j - 1)) \\
&\quad + (-1)^{\nu+1} P_\nu[\alpha\xi^{-l}]|_{\xi=1} \prod_{j=1}^{\nu} (2(h_H - l) + h_J - 1 - r - (h_H + h_J - l - j)) \Big) \\
&= \sum_{\nu=0}^{n} (-1)^\nu P_\nu[\alpha\xi^{-l}]|_{\xi=1} \prod_{j=1}^{\nu} (r - (h_H + h_J - l - j)) \\
&\quad + (-1)^{\nu+1}(-1)^\nu P_\nu[\alpha\xi^{-l}]|_{\xi=1} \prod_{j=1}^{\nu} (r - (h_H - l + j - 1)) = -\tilde{Q}(\alpha, r).
\end{aligned}$$

□

As a consequence all zeros of $\tilde{Q}(\alpha, r)$ are lying symmetrically to $h_H - l + (h_J - 1)/2$ in every branch $\alpha = u_0$.

We can now rewrite (12) in the form

$$Q(\alpha, r) = \left(D_J[\alpha\xi^{-l}]\xi^{r-l}\right) \xi^{-(h_H - l)}|_{\xi=1} \hat{P}(\alpha)\alpha + \tilde{Q}(\alpha, r)\hat{Q}(\alpha, r) \qquad (15)$$

with $\tilde{Q}(\alpha, r)$ symmetrical to $h_H - l + (h_J - 1)/2$ and degree h_J and with $\hat{Q}(\alpha, r)$ symmetrical to $(h_H - 1)/2$, degree $h_H - 2l$ and get the following corollary.

Corollary 1. If the degree of $\tilde{Q}(\alpha, r)$, i.e. h_J, is odd, then $r = h_H - l + (h_J - 1)/2$ is a zero of $\tilde{Q}(\alpha, r)$ in every branch $\alpha = u_0$. If $\alpha = u_0$ is a branch in $\delta H[u]$, i.e. $\hat{P}(u_0) = 0$, then $r = h_H$ is resonance in this branch, since $r = -1$ is one.

These symmetry conditions on the resonance polynomials developed above mean that if (roughly speaking) half the number of resonances is given the other half can be determined.

Example 4. *(Caudrey–Dodd–Gibbon–Sawada–Kotera II)*

$$\begin{aligned}
\Delta : u_t - J[u]\delta H[u] &= u_t - (u_{7x} + 42uu_{5x} + 84u_x u_{4x} + 126u_{2x}u_{3x} + 252u_x^3 \\
&\quad + 1512uu_x u_{2x} + 504u^2 u_{3x} + 2016u^3 u_x) \\
&= 0
\end{aligned}$$

$$H[u] = \tfrac{1}{2}u_{2x}^2 - 9uu_x^2 + 6u^4 \in \mathcal{H}_{-2}^{8} \qquad \delta H = u_{4x} + 18uu_{2x} + 9u_x^2 + 24u^3$$

$$J[u] = D_x^3 + 12D_x u + 12uD_x$$

$$\hat{P}(\alpha)\alpha = 24\left(\alpha^2 + 6\alpha + 5\right) = 24(\alpha + 1)(\alpha + 5)$$

$$\tilde{P}(\alpha) = \left(D_x^3 + 12\alpha D_x \xi^{-2} + 12\alpha\xi^{-2}D_x\right)\xi^{-6}|_{\xi=1} = -168(2 + \alpha)$$

$$\hat{Q}(-1, r) = (r+1)(r-3)(r-4)(r-8) \qquad \hat{Q}(-5, r) = (r+5)(r+1)(r-8)(r-12)$$

with $\hat{Q}(-1,r)$ and $\hat{Q}(-5,r)$ symmetrical to $(h_H-1)/2=\frac{7}{2}$

$$\tilde{Q}(-1,r)=(r-2)(r-7)(r-12) \qquad \tilde{Q}(-5,r)=(r+4)(r-7)(r-18)$$

with $\tilde{Q}(-1,r)$ and $\tilde{Q}(-5,r)$ symmetrical to $h_H-2+(3-1)/2=7$ with zero $r=7$.

The resonance polynomial in the branch $u_0=-2$ generated by $J[u]$ has the form

$$Q(-2,r)=(r+2)(r+1)(r-3)(r-4)(r-8)(r-9)(r-14)$$

with zero $r=14=2(h_H-1)$ and the other zeros lying symmetrically to $(h_H-1)/2=\frac{7}{2}$. (For the discussion of this case see the next section.)

5. Hamiltonian forms and Painlevé test

In this section we examine evolution equations with certain Hamiltonian forms. For some classes we show that they do not pass the Painlevé test right from the start and for others we develop formulae for all the possible principal resonance polynomials they can have, at most, in the Painlevé case. These formulae only depend on the skew-adjoint operator $J[u]$, the leading order l and the degree of homogeneity h_H of the conserved density $H[u]$.

Since we can choose $u_0=-1$ for one principal branch and since there are the two possibilities that either u_0 is a branch in $\delta H[u]$, i.e. $\hat{P}(-1)=0$ or one is generated by $J[u]$, i.e. $\tilde{P}(-1)=0$ we have (with (15)) to discuss the cases

I: $\hat{P}(-1)=0 \qquad Q(-1,r)=\tilde{Q}(-1,r)\hat{Q}(-1,r)$

and

II: $\tilde{P}(-1)=J[-\xi^{-l}]\xi^{-(h_H-l)}|_{\xi=1}=0$

$$Q(-1,r)=-\left(D_J[-\xi^{-l}]\xi^{r-l}\right)\xi^{-(h_H-l)}|_{\xi=1}\hat{P}(-1)+\tilde{Q}(-1,r)\hat{Q}(-1,r).$$

Further we can assume h_J to be odd.

5.1. For I

From corollary 1 we have $r=h_H-l+(h_J-1)/2$ and $r=h_H$ are zeros of $Q(-1,r)$ and with (10) we get $|\Delta r|\neq h_H+h_J-2l$, so that $r=l-(h_J+1)/2$, $r=2l-h_J$ and $r=h_H+h_J-2l-1$ (since $r=-1$ is resonance) are not zeros of $Q(-1,r)$ in the PPC case.

5.1.1. $h_J=1\Rightarrow J[u]=D_x$.
$l=1\Rightarrow \mathrm{o}(K)=h_H-1$:

$$\begin{aligned}Q(-1,r)&=\tilde{Q}(-1,r)\hat{Q}(-1,r)\\&=(r-(h_H-1))(r+1)\left(\prod_{i=1}^{h_H-4}(r-(1+i))\right)(r-h_H).\end{aligned} \tag{16}$$

Because of the symmetry of $\hat{Q}$ to $(h_H-1)/2$ and $r=0$, $r=1$ are not zeros in the PPC case, this is the only possible principal resonance polynomial.

Example 5. (*mKdV*)

$$\Delta : u_t - K[u] = u_t - (u_{3x} - 6u^2 u_x) = 0$$

$$H[u] = -\tfrac{1}{2}u_x^2 - \tfrac{1}{2}u^4 \in \mathcal{H}^4_{-1} \qquad \delta H = u_{2x} - 2u^3$$

$$J[u] = D_x \qquad Q(-1, r) = (r+1)(r-3)(r-4)$$

Example 6. (*Fordy–Gibbons*)

$$\Delta : u_t - (u_{5x} - 5u_x u_{3x} - 5u_{2x}^2 - 5u^2 u_{3x} - 20uu_x u_{2x} - 5u_x^3 + 5u^4 u_x) = 0 \quad (17)$$

$$H[u] = \tfrac{1}{2}u_{2x}^2 + \tfrac{5}{6}u_x^3 + \tfrac{5}{2}u^2 u_x^2 + \tfrac{1}{6}u^6 \in \mathcal{H}^6_{-1}$$

$$\delta H = u_{4x} - 5u_x u_{2x} - 5u^2 u_{2x} - 5uu_x^2 + u^5 \qquad J[u] = D_x$$

$$Q(-1, r) = (r+1)(r-2)(r-3)(r-5)(r-6).$$

$$l = 2 \Rightarrow \mathrm{o}(K) = h_H - 3 :$$

$$Q(-1, r) = \tilde{Q}(-1, r)\hat{Q}(-1, r)$$

$$= (r - (h_H - 2))(r+1)\left(\prod_{\substack{i \neq h_H - 5 \\ i \neq 2 \\ i=1}}^{h_H - 4} (r - (1+i))\right)(r - h_H). \quad (18)$$

Because of the symmetry of $\hat{Q}$ to $(h_H - 1)/2$ and $r = 0$, $r = 1$ and $r = 3$ are not zeros in the PPC case, this is the only possible principal resonance polynomial.

Example 7. (*KdV*)

$$\Delta : u_t - K[u] = u_t - (u_{3x} + 12uu_x) = 0 \quad (19)$$

$$H[u] = -\tfrac{1}{2}u_x^2 + 2u^3 \in \mathcal{H}^6_{-2} \qquad \delta H = u_{2x} + 6u^2$$

$$J[u] = D_x \qquad Q(-1, r) = (r+1)(r-4)(r-6).$$

Example 8. (*Second KdV*)

$$\Delta : u_t - (u_{5x} + 20uu_{3x} + 40u_x u_{2x} + 120u^2 u_x) = 0$$

$$H[u] = \tfrac{1}{2}u_{2x}^2 - 10uu_x^2 + 10u^4 \in \mathcal{H}^8_{-2} \qquad \delta H = u_{4x} + 20uu_{2x} + 10u_x^2 + 40u^3$$

$$J[u] = D_x \qquad Q(-1, r) = (r+1)(r-2)(r-5)(r-6)(r-8).$$

5.1.2. $h_J = 3$.
$l = 1$: There is *no* principal resonance polynomial in this case, because from corollary 1 we have $r = h_H$ is the zero of $Q(-1,r)$ and with (10) we get that $r = h_H$ (since $r = -1$ is resonance) cannot be the zero of $Q(-1,r)$ in the PPC case. Therefore no evolution equation of this form having the Painlevé property can exist.

$l = 2 \Rightarrow J[u] = D_x^3 + \beta u D_x + \beta D_x u \Rightarrow \mathrm{o}(K) = h_H - 1$: $\tilde{Q}(\alpha, r)$ is symmetrical to $h_H - 1$ and of degree 3, so we have $r = h_H - 1$ is resonance. $\hat{Q}(\alpha, r)$ is symmetrical to $(h_H - 1)/2$, degree $h_H - 4$, so $r = h_H$ is resonance. Since $|\Delta r| \neq h_H - 1$ therefore $r = 0$, $r = 1$ and $r = h_H - 2$ cannot be resonances in the PPC case.

We get $h_H - 4$ different possibilities for a principal resonance polynomial

$$Q(-1,r) = (r-(h_H-1+k))(r-(h_H-1))(r-(h_H-1-k))(r+1)$$
$$\times \left(\prod_{\substack{i=1 \\ i\neq k-1 \\ i\neq h_H-k-2}}^{h_H-4} (r-(1+i)) \right)(r-h_H) \qquad k = 2, \ldots, h_H - 3 \quad (20)$$

with $\beta = (k^2 - 1)/2$.

Example 9. *(Kaup–Kupershmidt)*

$$\Delta : u_t = u_{5x} + 15uu_{3x} + \tfrac{75}{2}u_x u_{2x} + 45u^2 u_x \qquad (21)$$
$$H[u] = -\tfrac{1}{2}u_x^2 + 2u^3 \in \mathcal{H}_{-2}^6 \qquad \delta H = u_{2x} + 6u^2 \qquad k = 2$$
$$J[u] = D_x^3 + \tfrac{3}{2}D_x u + \tfrac{3}{2}uD_x$$
$$Q(-1,r) = (r+1)(r-3)(r-5)(r-6)(r-7).$$

Example 10. *(Second KdV)*

$$\Delta : u_t - (u_{5x} + 20uu_{3x} + 40u_x u_{2x} + 120u^2 u_x) = 0 \qquad (22)$$
$$H[u] = -\tfrac{1}{2}u_x^2 + 2u^3 \in \mathcal{H}_{-2}^6 \qquad \delta H = u_{2x} + 6u^2$$
$$k = 3 \qquad J[u] = D_x^3 + 4D_x u + 4uD_x$$
$$Q(-1,r) = (r+1)(r-2)(r-5)(r-6)(r-8).$$

5.1.3. $h_J = 5$.
$l = 2$: There is *no* principal resonance polynomial in this case, because from corollary 1 we have $r = h_H$ is zero of $\tilde{Q}(-1,r)$ and we also get that $r = h_H$ is the zero of $\hat{Q}(-1,r)$. Therefore $r = h_H$ is a double zero of $Q(-1,r)$ and we do not have enough different positive resonances. As a consequence no evolution equation of this form having the Painlevé property can exist.

5.2. For II

5.2.1. $h_J = 1$. $J[u] = D_x$ does not generate a new branch; we have previously discussed this case.

5.2.2. $h_J = 3$. $l = 2 \Rightarrow J[u] = D_x^3 + \beta u D_x + \beta D_x u \Rightarrow \mathrm{o}(K) = h_H - 1$: Since $\tilde{P}(-1) = 0$ we get for β

$$\begin{aligned}\tilde{P}(-1) &= J[-\xi^{-2}]\xi^{-(h_H-2)}|_{\xi=1} \\ &= -(h_H-2)(h_H-1)h_H + \beta(h_H-2) + \beta h_H = 0 \Rightarrow \beta = \tfrac{1}{2}(h_H-2)h_H\end{aligned}$$

and for the principal resonance polynomial $Q(-1, r)$

$$\begin{aligned}Q(-1,r) &= -\tfrac{1}{2}(h_H-2)h_H(\xi^{r-2}D_x\xi^{-(h_H-2)} + D_x\xi^{r-h_H})|_{\xi=1}\hat{P}(-1) \\ &\quad + ((r-(h_H-2))(r-(h_H-1))(r-h_H) - \tfrac{1}{2}(h_H-2)h_H(r-(h_H-2)) \\ &\quad - \tfrac{1}{2}(h_H-2)h_H(r-h_H))\hat{Q}(-1,r) \\ &= -\tfrac{1}{2}(h_H-2)h_H(r-2(h_H-1))\hat{P}(-1) \\ &\quad + (r-(h_H-1))((r-(h_H-2))(r-h_H) - (h_H-2)h_H)\hat{Q}(-1,r) \\ &= (r-2(h_H-1))(-\tfrac{1}{2}(h_H-2)h_H\hat{P}(-1) + (r-(h_H-1))r\hat{Q}(-1,r)) \\ &=: (r-2(h_H-1))\bar{Q}(-1,r).\end{aligned}$$

$\bar{Q}(-1, r)$ is still symmetrical to $(h_H - 1)/2$, so $r = h_H$ must be a resonance (since $r = -1$ is one). With (10) we get $r = 1$, $r = h_H - 1$ (since $r = 2(h_H - 1)$ is one), $r = h_H - 2$ (since $r = -1$ is one) cannot be resonances in the PPC case.

Therefore

$$Q(-1,r) = (r-2(h_H-1))(r+1)\left(\prod_{i=1}^{h_H-4}(r-(1+i))\right)(r-h_H) \tag{23}$$

is the only possible principal resonance polynomial.

Example 11. (*KdV*)

$$\begin{aligned}&\Delta : u_t - K[u] = u_t - (u_{3x} + 12uu_x) = 0 \\ &H[u] = \tfrac{1}{2}u^2 \in \mathcal{H}^4_{-2} \qquad \delta H = u \qquad J[u] = D_x^3 + 4uD_x + 4D_xu \\ &Q(-1,r) = (r+1)(r-4)(r-6).\end{aligned}$$

Example 12. (*Caudrey–Dodd–Gibbon–Sawada–Kotera*)

$$\begin{aligned}&\Delta : u_t - (u_{5x} + 30uu_{3x} + 30u_xu_{2x} + 180u^2u_x) = 0 \\ &H[u] = -\tfrac{1}{2}u_x^2 + u^3 \in \mathcal{H}^6_{-2} \qquad \delta H = u_{2x} + 3u^2 \\ &J[u] = D_x^3 + 12D_xu + 12uD_x \\ &Q(-1,r) = (r+1)(r-2)(r-3)(r-6)(r-10).\end{aligned}$$

6. A construction algorithm

From the above results we can formulate the following construction algorithm for evolution equations in one of the previously discussed Hamiltonian forms:

- Choose a leading order l and a degree of homogeneity h for a starting conserved density.
- Construct this density out of all irreducible monomials with degree of homogeneity h and free coefficients.
- Execute the variational derivative and choose the form of the skew-adjoint operator $J[u]$.
- Construct the possible resonance pattern(s) and adapt some (all) of the free coefficients to get a $K[u]$ with the appropriate resonances.
- Perform the Painlevé test in the principal branch and adapt the rest of the free parameters.
- If there are still free coefficients left, look for other branches passing the Painlevé test.
- If you end up with a PPC case perform the whole Painlevé test and construct and examine the most general inhomogeneous case in an analogous way.

Example 13.

(i) Case I with $l = 2$, $h_H = 6$ and $h_J = 3$:

$$H[u] = -\tfrac{1}{2}u_x^2 + \gamma_1 u^3 \in \mathcal{H}^6_{-2} \qquad \delta H = u_{2x} + 3\gamma_1 u^2$$

$$J[u] = D_x^3 + \beta u D_x + \beta D_x u \qquad \text{with } \beta = (k^2-1)/2 \text{ and } k = 2, 3$$

$$\Delta : u_t - (u_{5x} + 2(\beta + 3\gamma_1)uu_{3x} + (\beta + 18\gamma_1)u_x u_{2x} + 15\gamma_1\beta u^2 u_x) = 0.$$

The principal resonance polynomial for this evolution equation exhibiting the free parameter γ_1 is

$$Q(-1, r) = r^5 - 20r^4 - (6\gamma_1 + 2\beta - 155)r^3 + (20\beta + 90\gamma_1 - 580)r^2$$
$$- (444\gamma_1 - 15\gamma_1\beta + 68\beta - 1044)r - 90\gamma_1\beta + 720\gamma_1 + 120\beta - 720.$$

With (20) we have the following two possibilities of principal resonance patterns in the PPC case

k	β	r_1	r_2	r_3	r_4	r_5	γ_1
2	$\frac{3}{2}$	-1	3	5	6	7	2
3	4	-1	2	5	6	8	2

Since all parameters are determined, we can perform the Painlevé test in the well known way and we end up with the cases $k = 2$ (Kaup–Kupershmidt (21)) and $k = 3$ (second KdV (22)) passing this test.

The next step of the algorithm is to extend the homogeneous equation to the most general inhomogeneous one. In this example we will discuss the second KdV equation, i.e.

$$\Delta : u_t - K_2[u] = u_t - (u_{5x} + 20uu_{3x} + 40u_x u_{2x} + 120u^2 u_x) = 0.$$

The general inhomogeneous irreducible conserved density H_{inh} is

$$H_{\text{inh}}[u] = -\tfrac{1}{2}u_x^2 + 2u^3 + \gamma_2 u^2$$

and the corresponding inhomogeneous skew-adjoint operator J_{inh}

$$J_{\text{inh}}[u] = D_x^3 + 4uD_x + 4D_x u + \gamma_3 D_x.$$

Performing the Painlevé test in the principal branch, we end up, that with

$$\gamma_2 = \tfrac{1}{2}\alpha_1 \qquad \gamma_3 = 20\alpha$$

the following inhomogeneous evolution equation is a PPC

$$\Delta_{\text{inh}} : u_t - K_2[u+\alpha] - \alpha_1 K_1[u+\alpha] - \alpha(8\alpha_1 - 120\alpha)u_x = 0 \qquad \alpha, \alpha_1 \in \mathbb{R} \text{ arbitrary}$$

where $K_1[u]$ denotes the right-hand side of the KdV equation (19).

(ii) Case I with $l = 2$, $h_H = 8$ and $h_J = 3$:

$$H[u] = \tfrac{1}{2}u_{2x}^2 + \gamma_1 u u_x^2 + \gamma_2 u^4 \in \mathcal{H}_{-2}^8 \qquad \delta H = u_{4x} - 2\gamma_1 u u_{2x} - \gamma_1 u_x^2 + 4\gamma_2 u^3$$

$$J[u] = D_x^3 + \beta u D_x + \beta D_x u \qquad \text{with } \beta = (k^2 - 1)/2 \text{ and } k = 2, \ldots, 5$$

$$\Delta : u_t - (u_{7x} + 2(\beta - \gamma_1)uu_{5x} + (\beta - 8\gamma_1)u_x u_{4x} - 14\gamma_1 u_{2x}u_{3x} + 4(3\gamma_2 - \gamma_1\beta)u^2 u_{3x}$$
$$+ (72\gamma_2 - 10\gamma_1\beta)uu_x u_{2x} + (24\gamma_2 - \gamma_1\beta)u_x^3 + 28\gamma_2\beta u^3 u_x) = 0.$$

The principal resonance polynomial for this evolution equation exhibiting free parameters γ_1, γ_2 is

$$Q(-1,r) = r^7 - 35r^6 + (2\gamma_1 - 2\beta + 511)r^5 + 7(6\beta - 8\gamma_1 - 575)r^4$$
$$+ 2(309\gamma 1 - 2\gamma_1\beta + 6\gamma_2 - 169\beta + 9212)r^3$$
$$+ 14(4\gamma_1\beta - 242\gamma_1 - 18\gamma_2 + 93\beta - 3490)r^2$$
$$+ 4(2344\gamma_1 - 69\gamma_1\beta - 7\gamma_2\beta + 438\gamma_2 - 629\beta + 17316)r$$
$$+ 224(3\gamma_1\beta - 48\gamma_1 + \gamma_2\beta - 18\gamma_2 + 15\beta - 180).$$

With (20) we have the following four possibilities for the principal resonance patterns in the PPC case

k	β	r_1	r_2	r_3	r_4	r_5	r_6	r_7	γ_1	γ_2
2	$\frac{3}{2}$	−1	3	4	5	7	8	9	−9	6
3	4	−1	2	4	5	7	8	10	−10	10
4	$\frac{15}{2}$	−1	2	3	5	7	8	11	−10	10
5	12	−1	2	3	4	7	8	12	−9	6

Since all parameters are determined, we can perform the Painlevé test in the well known way and we end up with the cases $k = 2$ (Kaup–Kupershmidt II), $k = 3$ (third KdV) and $k = 5$ (Caudrey–Dodd–Gibbon–Sawada–Kotera II) passing this test, whereas for $k = 4$ in the principal branch $u_0 = -1$ the compatibility condition at $r = 5$ cannot be fulfilled. Therefore this equation cannot have the Painlevé property and is of no further interest here.

The next step of the algorithm is to extend the homogeneous equation to the most general inhomogeneous one. In this example we will discuss the Kaup–Kupershmidt II equation, i.e.

$$\begin{aligned}\Delta : u_t - K_{\mathrm{II}}[u] = u_t - (&u_{7x} + 21uu_{5x} + \tfrac{147}{2}u_x u_{4x} + 126u_{2x}u_{3x} + 126u^2 u_{3x}\\ &+ 567uu_x u_{2x} + \tfrac{315}{2}u_x^3 + 252u^3 u_x) = 0.\end{aligned} \tag{24}$$

The general inhomogeneous irreducible conserved density H_{inh} is

$$H_{\mathrm{inh}}[u] = \tfrac{1}{2}u_{2x}^2 - 9uu_x^2 + 6u^4 + \gamma_3 u^3 + \gamma_4 u_x^2 + \gamma_5 u^2$$

and the corresponding inhomogeneous skew-adjoint operator J_{inh}

$$J_{\mathrm{inh}}[u] = D_x^3 + \tfrac{3}{2}uD_x + \tfrac{3}{2}D_x u + \gamma_6 D_x.$$

Performing the Painlevé test in the principal branch, we end up that with

$$\gamma_3 = 24\alpha + 2\alpha_1 \qquad \gamma_4 = -9\alpha - \tfrac{1}{2}\alpha_1 \qquad \gamma_5 = 6\alpha(6\alpha + \alpha_1) \qquad \gamma_6 = 3\alpha$$

the following inhomogeneous evolution equation is a PPC

$$\Delta_{\mathrm{inh}} : u_t - K_{\mathrm{II}}[u+\alpha] - \alpha_1 K_{\mathrm{I}}[u+\alpha] - 9\alpha^2(4\alpha+\alpha_1)u_x = 0 \qquad \alpha, \alpha_1 \in \mathbb{R} \text{ arbitrary} \tag{25}$$

where $K_{\mathrm{I}}[u]$ denotes the right-hand side of the Kaup–Kupershmidt equation (21).

Remark. The phenomenon that the inhomogeneous PPCs are shifts in the homogeneous equations plus their infinitesimal symmetries lying below in degree of homogeneity times arbitrary constants, i.e. are of the form $\Delta : u_t - K[u + \alpha] - \alpha_1 K_1[u + \alpha] - \cdots - \alpha_n K_n[u + \alpha] - f(\alpha, \alpha_1, \ldots, \alpha_n)u_x = 0$ with $K[u] \in \mathcal{H}^h_{-l}$ and $K_i[u] \in \mathcal{H}^{h_i}_{-l}, h_i < h$ infinitesimal symmetries of $\Delta : u_t - K[u] = 0$, can always be observed and will be object of further investigations.

(iii) Case I with $l = 1$, $h_H = 6$ and $h_J = 1$:

$$H[u] = \tfrac{1}{2}u_{2x}^2 + \gamma_1 u_x^3 + \gamma_2 u^2 u_x^2 + \gamma_3 u^6 \mathcal{H}^6_{-1} \qquad J[u] = D_x$$

$$\begin{aligned}\Delta : u_t - (&u_{5x} - 6\gamma_1 u_x u_{3x} - 6\gamma_1 u_{2x}^2 - 2\gamma_2 u_x^3 - 8\gamma_2 uu_x u_{2x}\\ &- 2\gamma_2 u^2 u_{3x} + 30\gamma_3 u^4 u_x) = 0.\end{aligned}$$

In the PPC case, Δ has to have (with (16)) the principal resonance polynomial

$$Q(-1, r) = (r + 1)(r - 2)(r - 3)(r - 5)(r - 6). \tag{26}$$

Adapting the free parameters according to (26) and performing the test in the principal branch, we end up with the PPC

$$\begin{aligned}\Delta : u_t - \tilde{K}[u] - 6\gamma_1 \hat{K}[u] &= u_t - \left(u_{5x} - 10u_x^3 - 40uu_xu_{2x} - 10u^2u_{3x} + 30u^4u_x\right) \\ &\quad - 6\gamma_1\left(-u_xu_{3x} - u_{2x}^2 + u_x^3 + 4uu_xu_{2x} + u^2u_{3x} - 5u^4u_x\right) \\ &= 0 \end{aligned} \tag{27}$$

and $\gamma_2 = -(3\gamma_1 - 5)$, $\gamma_3 = -(\gamma_1 - 1)$.

Remark. The differential polynomial $\hat{K}$ in (27) is called *r-degenerate* [15], since its resonance polynomial in the branch $u_0 = -1$ vanishes identically.

If we now demand that (27) should have another principal branch $u_{0,2} \neq -1$, and since there is only one possible resonance pattern, we have to solve the equation

$$\tilde{K}[u] + \hat{\gamma}_1 \hat{K}[u] = \frac{1}{a}\left(\tilde{K}[au] + \tilde{\gamma}_1 \hat{K}[au]\right) \qquad a \neq 1$$

and get the solutions

	a	$\hat{\gamma}_1$	$\tilde{\gamma}_1$	$u_{0,2}$
(i)	-1	0	0	1
(ii)	$-\frac{1}{2}$	$\frac{5}{6}$	$-\frac{5}{3}$	2
(iii)	-2	$-\frac{5}{3}$	$\frac{5}{6}$	$\frac{1}{2}$

Since cases (ii) and (iii) are symmetrical, we only have the two different cases: the first is the second *mKdV*, Δ: $u_t - (u_{5x} - 10u_x^3 - 40uu_xu_{2x} - 10u^2u_{3x} + 30u^4u_x) = 0$; and the second with $\gamma_1 = \frac{5}{6}$ is (17), the Fordy–Gibbons equation.

The author has implemented this algorithm in a REDUCE package, which is published in [16].

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